Modeling and Simulation of a Model Rocket Driven by Pressurized Water as a Function of Start Parameter Variations

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1 Introduction

A rocket is a good example to illustrate the laws of momentum conservation and their consequences. An emanating fluid out of the rocket body drives the rocket. The acceleration of the rocket body is mainly a function of the mass and relative velocity of this fluid. Space rockets utilize hot combustion gases as driving fluid, but any other gaseous or liquid fluids are principally eligible, too. It depends on the power required.

Small rockets are available for teaching purposes, where the driving fluid is pressurized water. The water exits the rocket body through a nozzle with high velocity. The rocket velocity and final travel height is mainly a function of start pressure and start water mass. This kind of rocket was subject of the investigation described in this report. The task was to provide a theoretically reasoned estimation of the rocket velocity and rocket travel height versus time. The investigation was focused on the time period between the rocket launch and when no water is left in the rocket, so when the rocket propulsion stopped. Basically two mechanical and one thermodynamic law describe the process. Newton’s law for bodies with changing mass provides information about the rocket movement in dependence of the current rocket mass, change of mass and relative exit velocity of the water. Bernoulli’s law for incompressible fluids describes the emanating mass flow out of the rocket body in time. The change of air pressure inside of the rocket body as a function of water volume enclosed can be described with a formulation for a polytropic change of state. Chapter 2 provides an overview of the mathematical description of the problem and its numerical solution. Chapter 3 provides the results obtained.

2 Mathematical Description

2.1 Rocket Body Velocity

Newton’s law for a body with changing mass determines the movement of the rocket body. According to [1], this law can be posted to:

$$m\dot{\vec{v}} = m\vec{g} + \dot{m}\vec{v}_{rel} \tag{2.1-1}$$

m	Rocket Mass, Function of Time
$\dot{m}$	Change of Rocket Mass in Time (Mass Flux)
$\dot{\vec{v}}$	Rocket Acceleration (Derivation of Rocket Velocity in Time)
$\vec{v}_{rel}$	Emanating Water Velocity Relative to Rocket Body
g	Gravitational Acceleration, 9.81 m/s^2

Eq. (2.1-1) is a vector equation. As considering two dimensions does not include more physics or an additional numerical problem we want to focus on one dimensional description. The extension to two dimensions is straightforward and can be done easily. With the assumption that the rocket and the exiting water only move in positive or negative *z*-direction respectively, one can write:

$$\begin{pmatrix} 0 \\ 0 \\ m\dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \dot{m}\begin{pmatrix} 0 \\ 0 \\ -v_{rel} \end{pmatrix} \tag{2.1-2}$$

This equation suggests considering only the *z*-direction. The friction between the moving rocket body and the non-moving air molecules can be described by using a constant drag coefficient in combination with the theoretical kinetic energy of the passing air, Eq. (2.1-3).

$$m\dot{v} = -mg - \dot{m}v_{rel} - \frac{1}{2}\rho_{Air} v^2 c_{wR} A_{\max R} \tag{2.1-3}$$

ρ_{Air}	Air Density, 1.2 kg/m^3
c_{wR}	Drag coefficient rocket body
$A_{\max R}$	Max. Rocket Body Cross Section Area in Flight Direction

The mass of the rocket changes with time due to the emanating water and can be expressed with the relative velocity of the water, Eq (2.1-4).

$$m = m_0 - \rho_w A_D \int_0^t v_{rel} dt \tag{2.1-4}$$

m_0	Rocket Start Mass, Sum of Start Water Mass and Rocket Body Mass
ρ_w	Water Density, 1000 kg/m^3
A_D	Nozzle Cross Section

The relative water velocity is equated for the nozzle cross section A_D. The mass flux can easily be calculated with Eq. (2.1-4). With this, Eq. (2.1-3) changes to:

$$\dot{v} = -g + \frac{v_{rel}^2}{x_0 - \int_0^t v_{rel} dt} - \frac{1}{2} c_{wR} \frac{\rho_{Air}}{\rho_W} \frac{A_{\max R}}{A_D} \frac{v^2}{x_0 - \int_0^t v_{rel} dt} \tag{2.1-5}$$

x_0	Parameter Definition ‘Start Length’, $m_0/(\rho_W A_D)$

One has to gain further information about the relative water velocity to be able to provide a numerical solution for this differential equation. This will be done in the following chapter.

2.2 Relative Water Velocity

The best approximation of the relative water velocity implies considering the movement of the rocket body as well, since the accelerated rocket body is a non-inertial system. Unfortunately, this complicates the mathematical description tremendously. Disregarding the rocket body velocity enables to use Bernoulli's law for a laminar fluid flow for describing the relative water velocity. The real water flow is highly turbulent though, reducing the accuracy of the predictions obtained. Bernoulli's law for an unsteady stream line from the water surface within the rocket body (Position 1, compare Fig. 2.2-I) leading to the exit of the nozzle (Position 2) can be stated as [2]:

$$\rho_W \int_1^2 \frac{\partial v_{rel}(s,t)}{\partial t} ds + \frac{1}{2}\rho_W (v_{vrel2}^2 - v_{rel1}^2) + p_2 - p_1 + \rho_W g (z_2 - z_1) = -\frac{1}{2}\rho_W c_{WN} v_{rel2}^2 \qquad (2.2\text{-}1)$$

c_{WN} Friction Coefficient for Nozzle

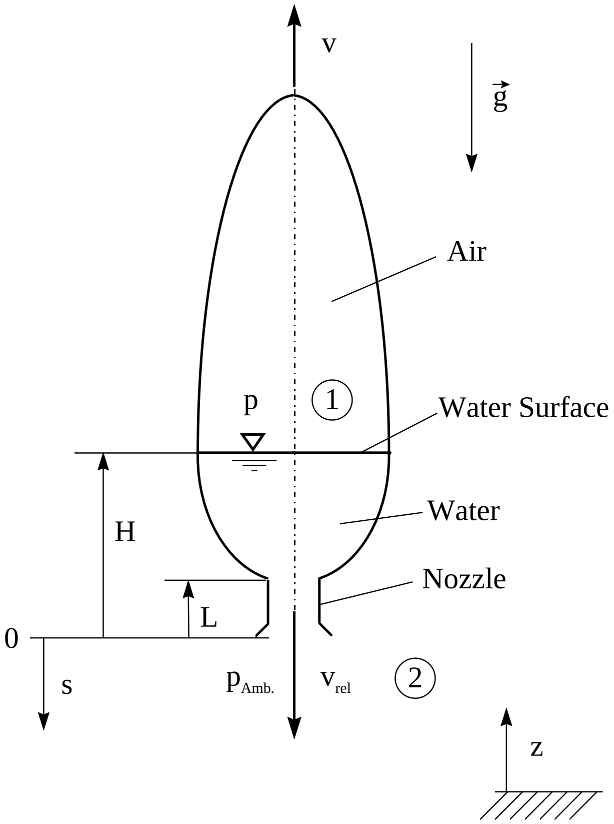

Fig. 2.2-I: Sketch of the Rocket

Equation (2.2-1) is valid for an incompressible fluid. The right side of this equation considers the fluid friction within the nozzle, represented by a kinetic pressure drop. The friction coefficient is mainly a function of the nozzle diameter and the material used and is assumed to be constant. The geostatic height difference between the water surface within the rocket body and the nozzle exit can be neglected, still disregarding the movement of the rocket body. The pressure p_1 acts upon the water surface within the rocket and changes during the emanating of the water, since the air is expanding. The pressure p_2 acting at the nozzle is identical with the atmospheric ambient pressure p_a and therefore constant. As the relative water velocity at the nozzle exit is needed, all other relative velocities within the rocket body have to be related to the velocity within the nozzle. This can be done using the law of mass conservation for an incompressible fluid, Eq. (2.2-2).

$$v_{rel}(s) = \frac{A_D}{A_S} v_{rel} \tag{2.2-2}$$

$v_{rel}(s)$	Relative Water Velocity at Position *s*
v_{rel}	Relative Water Velocity at Nozzle
A_S	Flow Cross Section at Position *s*

With these remarks made, one finds for Eq. (2.2-1):

$$\rho_W \dot{v}_{rel} \int_{-H}^{0} \frac{A_D}{A_S} ds + \frac{1}{2}\rho_W v_{rel}^2 (1 - \left(\frac{A_D}{A_H}\right)^2) + p_a - p = -\frac{1}{2}\rho_W c_{WN} v_{rel}^2 \tag{2.2-3}$$

A_H	Flow Cross Section at Position -*H*

The integration limits in Eq. (2.2-3) assume a coordinate system with the origin at the nozzle exit and negative values for positions within the rocket body, compare Fig. 2.2-I. Rearranging gives:

$$\dot{v}_{rel} = \frac{1}{\rho_W \int_{-H}^{0} \frac{A_D}{A_S} ds} (p - p_a + \frac{1}{2}\rho_W v_{rel}^2 (\left(\frac{A_D}{A_H}\right)^2 - c_{WN} - 1)) \tag{2.2-4}$$

Still unknown in this equation are the pressure and the current water height *H* within the rocket body and an expression of the flow cross section at any position on the streamline. The rocket body can be described as a rotation of a third grade polynomial and a cylinder of length *L* for the nozzle. With this, the radius of the rocket body can be approximated to:

$$R = as^3 + bs^2 + cs + d\,,\ s \le -L \tag{2.2-5}$$

$$R = 0.5D\,,\ -L \le s \le 0 \ \text{(Nozzle)} \tag{2.2-6}$$

R	Radius of Flow Cross Section at Position *s*
D	Nozzle Diameter

a,b,c,d	Polynomial Coefficients, Determined with Rocket Body Shape

The found polynomial is shown in Fig. 2.2-II:

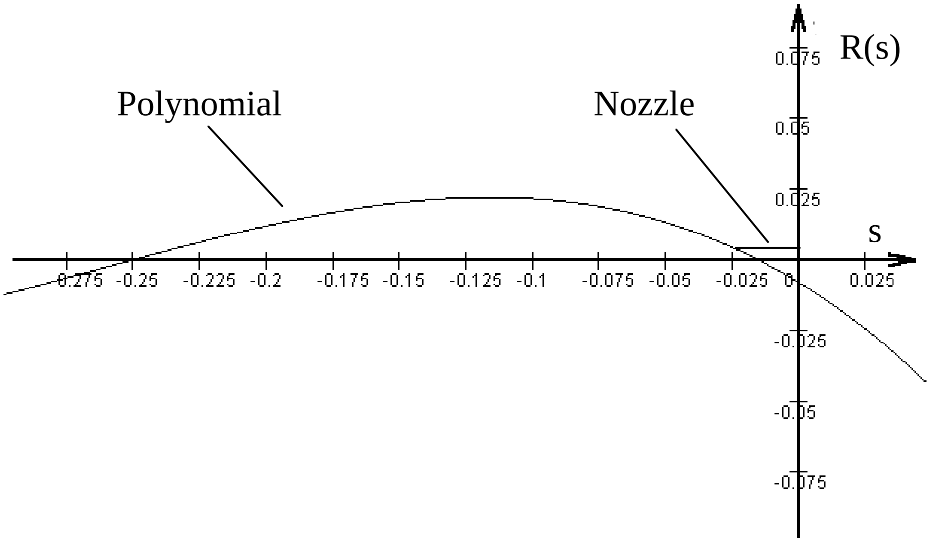

Fig. 2.2-II: Rocket Body Describing Polynomial

The cross section area is with Equation 2.2-5 and 2.2-6:

$$A_S = \pi R^2 \, , R = f(s) \; s \tag{2.2-7}$$

The pressure and the water volume within the rocket are related to each other. Reducing the water volume reduces the pressure as well, since the enclosed air is expanding when the water is exiting the rocket body. This expansion process can be described with an expression for a polytropic change of the thermodynamic state. The following expression is valid:

$$p = CV^{-n} \tag{2.2-8}$$

p	Pressure of the Enclosed Air
V	Volume of the Enclosed Air
C	Constant, Determined by Start Pressure And Start Volume
n	Constant Polytropic Exponent

The enclosed air volume is determined by the current water volume within the rocket body and can be stated as:

$$V = V_0 - V_W \tag{2.2-9}$$

V_0	Total Rocket Body Volume
V_W	Current Water Volume

The current water volume is a function of the current water height and can be approximated using expression (2.2-5), (2.2-6) and (2.2-7):

$$V_W = \int_{-H}^{0} A_S ds \quad = \pi \int_{-H}^{-L} \left(as^3 + bs^2 + cs + d \right)^2 ds + A_D L \, , \quad -H \leq -L \tag{2.2-10}$$

$$= A_D H \, , \quad -L \leq -H \leq 0 \tag{2.2-11}$$

The equations presented in this chapter build together with Eq. (2.1-5) an equation system, which can be solved numerically. The required algorithm and additional equations are presented in the following chapter.

2.3 Numerical Solution Algorithm

A numerical solution algorithm is mainly required to solve the differential equations Eq. (2.1-5) and (2.2-4). Both are of the form

$$\dot{\phi} = f(\phi^2) \tag{2.3-1}$$

ϕ	General Parameter

Equation (2.3-1) can be transferred into a numerically solvable equation by transforming the time derivation into a time difference quotient, Eq. (2.3-2).

$$\dot{\phi} = \lim_{\Delta t \to 0} \frac{\phi_{t+\Delta t} - \phi_t}{\Delta t} \tag{2.3-2}$$

Δt	Time Step

This suggests the following numerical form for Eq. (2.3-1):

$$\phi_{t+\Delta t} = \phi_t + \Delta t \cdot f(\phi_t^2) \tag{2.3-3}$$

This is a fully explicit scheme. The numerical solution algorithm is an iteration process, where a new value for the general parameter after the time step *Δt* is equated with the old value for the general parameter and so forth. The time step in the numerical applications has to be set by the user and is therefore a small, but finite number. Decreasing the time step enhances the accuracy but enhances the computing time as well. Enhancing the time step reduces the computing time but worsens the accuracy and might cause divergence of Eq. (2.3-3). The fix point iteration character of this equation (compare Leibniz-criteria) reasons this behavior. It is apparent that a compromise has to be found. Considering the rocket velocity in the Bernoulli equation, Eq. (2.2-1), would create a coupling between this equation and Newton's equation, Eq. (2.1.1). The numerical solution process would involve an iteration process to reach convergence for each time step, the numerical scheme would therefore be implicit and more expensive.

For Eq. (2.1-5), the general parameter is identical with the total rocket body velocity v, whereas it is the relative water velocity v_{rel} for Eq. (2.2-4). The missing connection to solve this equation system is the link between the time step and the change in water height. As a matter of mass conservation one can state:

$$\Delta h = \frac{A_D}{A_H} v_{rel} \Delta t \tag{2.3-4}$$

Δh	Change in Water Height

This results into the following summarizing table:

Time	**Water Height**	**General Parameter**	**Water Velocity**	**Rocket Velocity**
T	H	ϕ	v_{rel}	v
$T+\Delta t$	$H+\Delta h$	ϕ^{*}	v_{rel}^{*}	v^{*}

Table 2.3-I: Numerical Parameter Iteration

The abort criterion for the iteration process is reached when the water height *H* reaches zero, meaning no water is left in the rocket body. The movement of the rocket as a ballistic body upon zero water level in the rocket is easy to solve and was not considered. The propulsion due to air mass flow out of the rocket was neglected, since the effect is small. This case occurs when the inner rocket air pressure is higher than ambient pressure after water propulsion is finished.

The acceleration integral in Eq. (2.2-4) is computed numerically with Eq. (2.3-5) for each time step as well, using expression Eq. (2.2-5) – (2.2-7).

$$I_{ACC} = \int_{-H}^{0} \frac{A_D}{A_S} ds \approx \sum_{-H}^{0} \frac{A_D}{A_S} \Delta s \qquad (2.3\text{-}5)$$

The program code is written in Visual Basic, in combination with an Excel sheet for parameter input. Data plots were accomplished with Tecplot. The complete Visual Basic code can be found in Appendix B.

3 Results

3.1 Preliminary Remarks

Table 3.1-I provide all parameters used, which were not changed during the calculations. The polynomial coefficients a, b, c, d to describe the rocket body belong to this group as well as ambient pressure and the gravitational acceleration.

Pol. Coefficient a	-6.032	[1/m^2]	Total Rocket Volume	200	[cc]
Pol. Coefficient b	-3.682	[1/m]	Ambient Pressure	1013	[mbar]
Pol. Coefficient c	-0.6108	[-]	Water Density	1.2	[kg/m^3]
Pol. Coefficient d	-8.72e-3	[m]	Air Density	1000	kg/m^3
Nozzle Diameter D	0.025	[m]	Gravitational Acc.	9.81	m/s^2
Nozzle Length L	0.022	[m]	Rocket Drag Coefficient	0.4	[-]

Table 3.1-I: Constant Parameters During All Calculations

Besides parameters with a clear influence on the rocket behavior (start pressure, start water mass) especially those parameters were varied, where only assumptions about the right value could be made (e.g. nozzle friction coefficient).

The standard configuration to create a basis for comparison for all other variations is summarized in Table 3.1-II. In one single parameter investigation only one parameter was changed, all others were kept constant. The standard case is represented as a black solid line in all plots.

Start Pressure	3	[bar]	Nozzle Drag Coefficient	0.3	[-]
Start Water Height	0.116	[m]	Polytropic Exponent	1.3	[-]
Start Water Weight	0.086	[kg]	Time Step	0.8e-4	[s]

Table 3.1-II: Standard Configuration

3.2 Start Pressure Variation

The start pressure was varied in 1-bar steps from 3 to 7 bar.

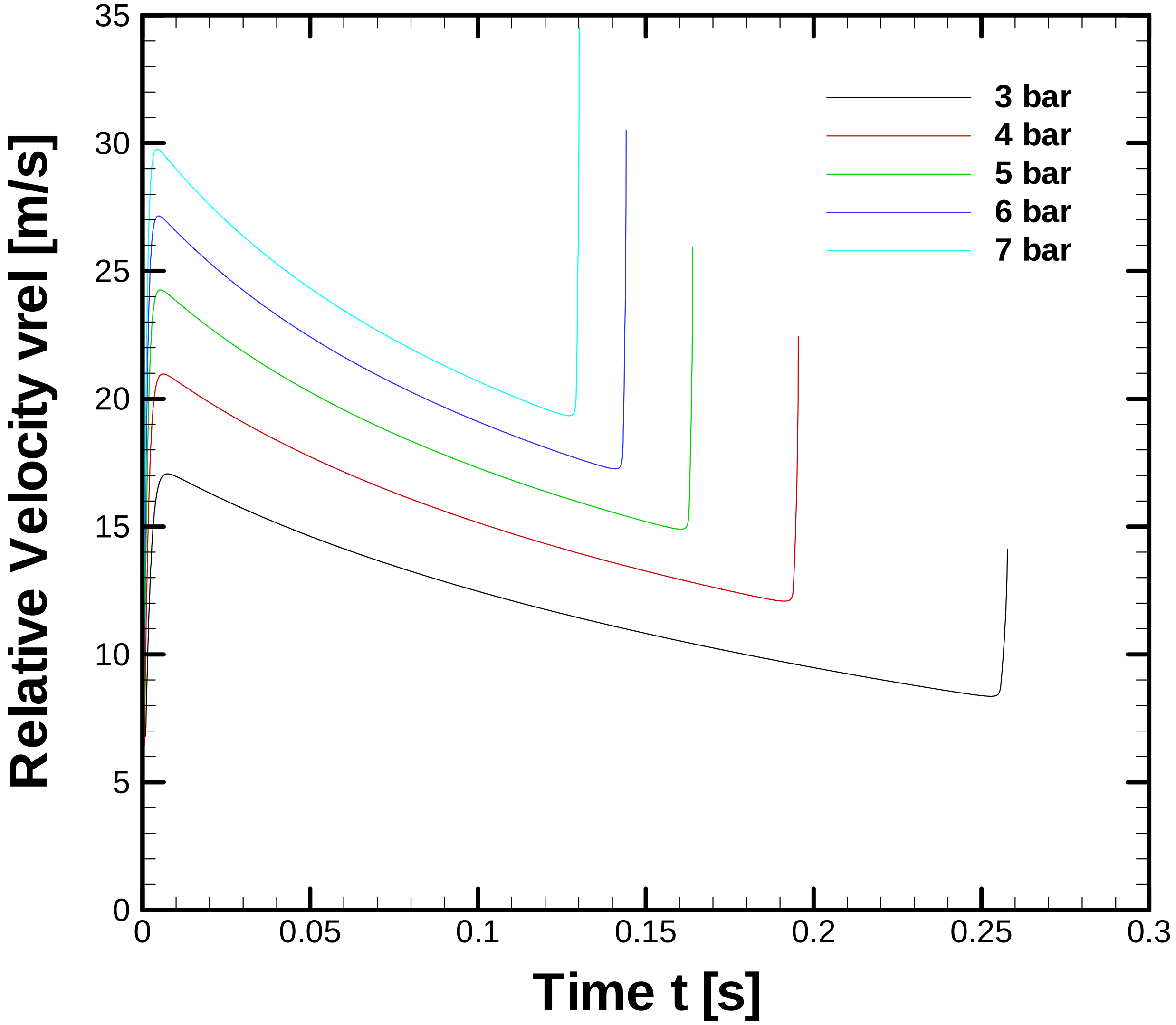

Fig. 3.2-I: Relative Water Velocity versus time as function of start pressure

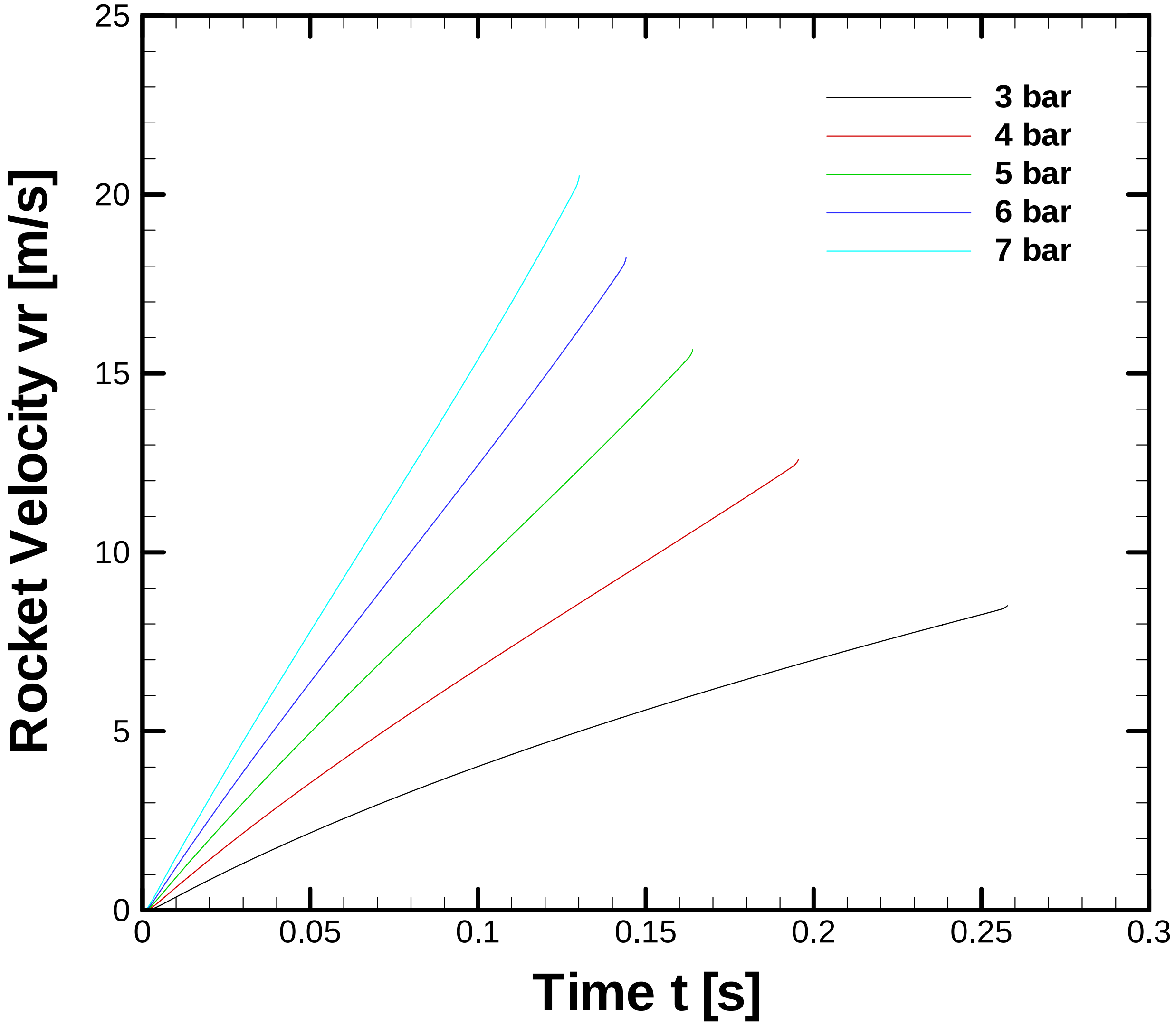

Fig. 3.2-II: Rocket Velocity versus time as function of start pressure

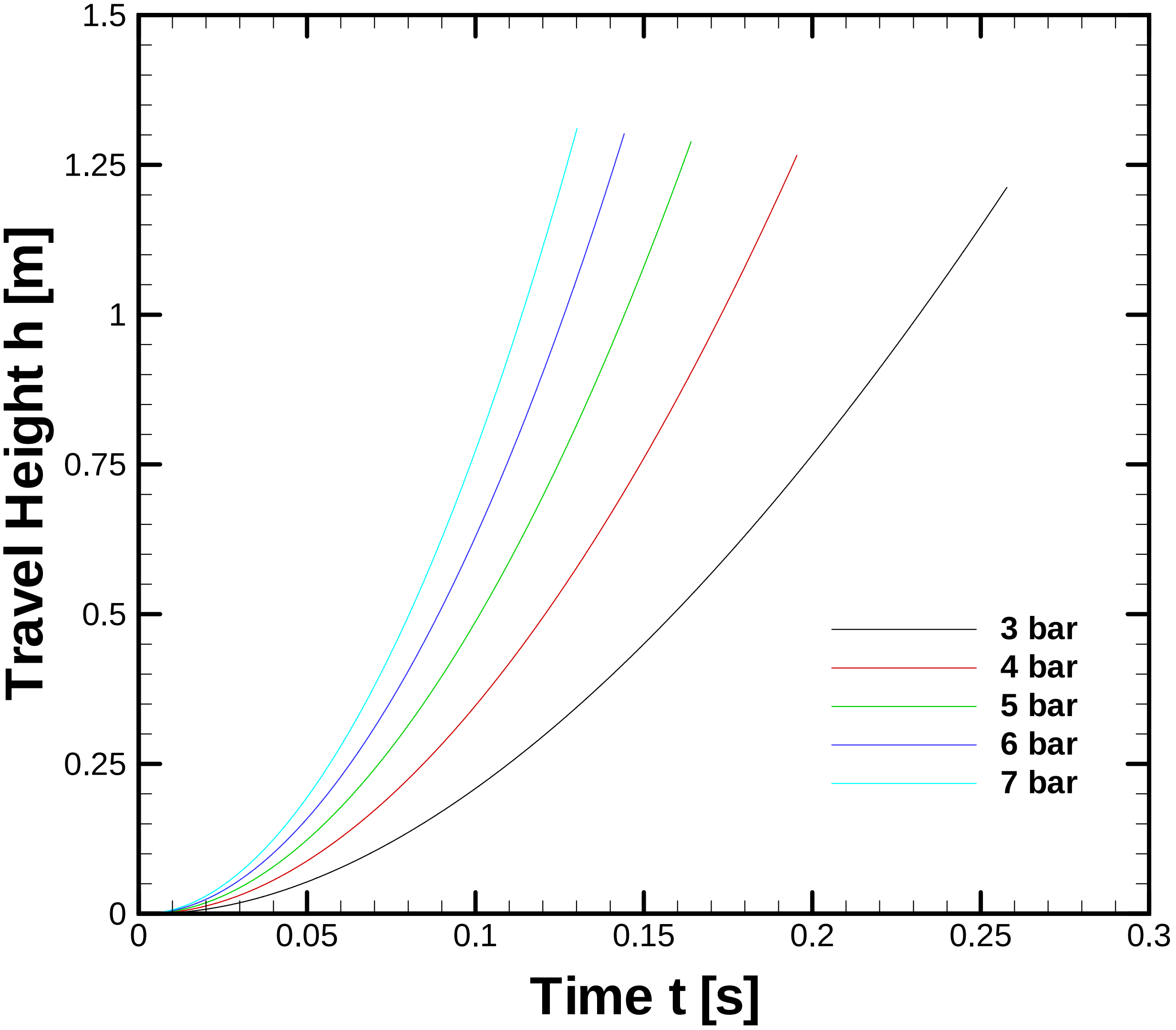

Fig. 3.2-III: Travel height versus time as function of start pressure

3.3 Start Water Mass Variation

The start water mass was varied from 65e-3 to 100e-3 kg.

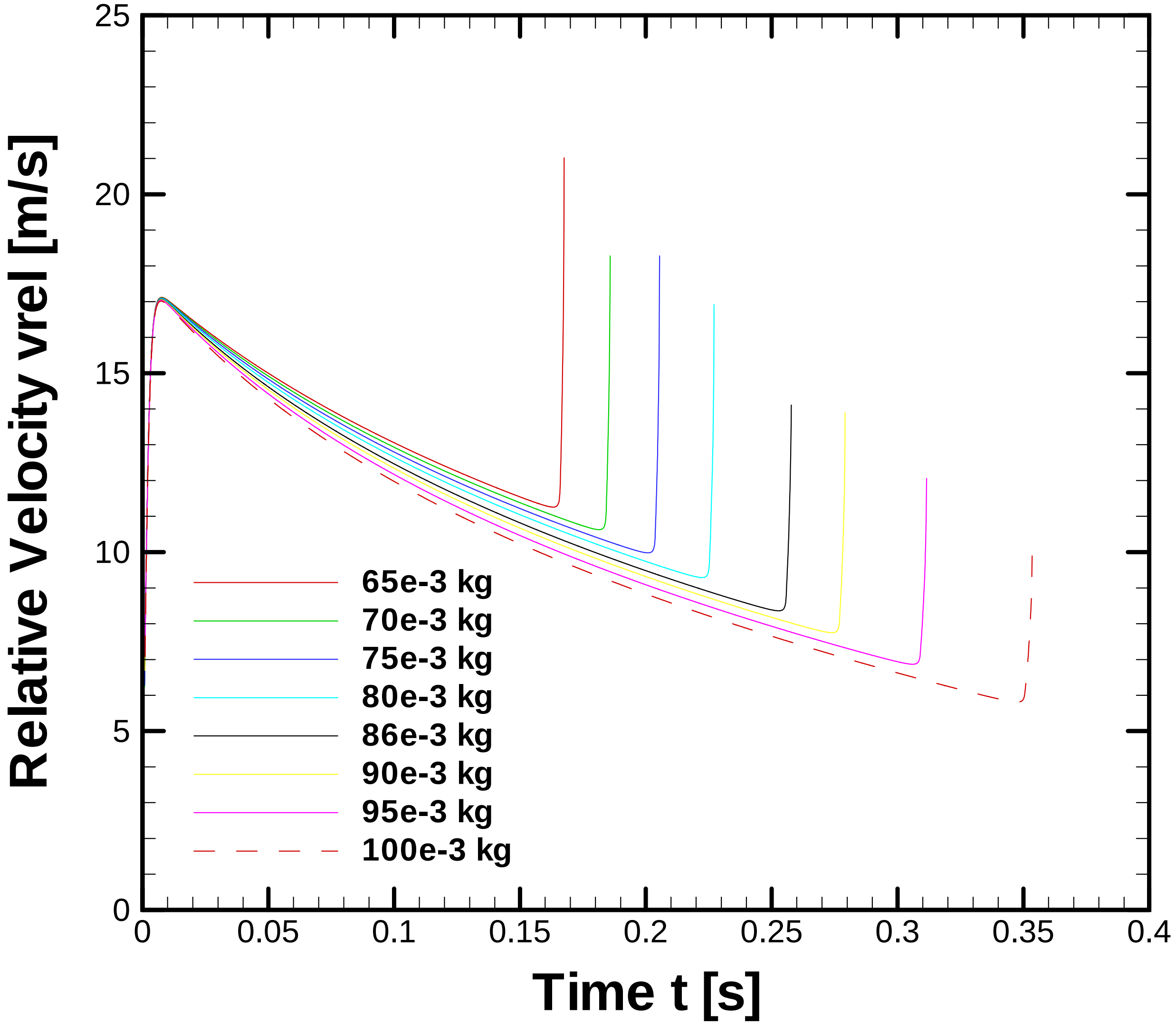

Fig. 3.3-I: Relative Water Velocity versus time as function of start water mass

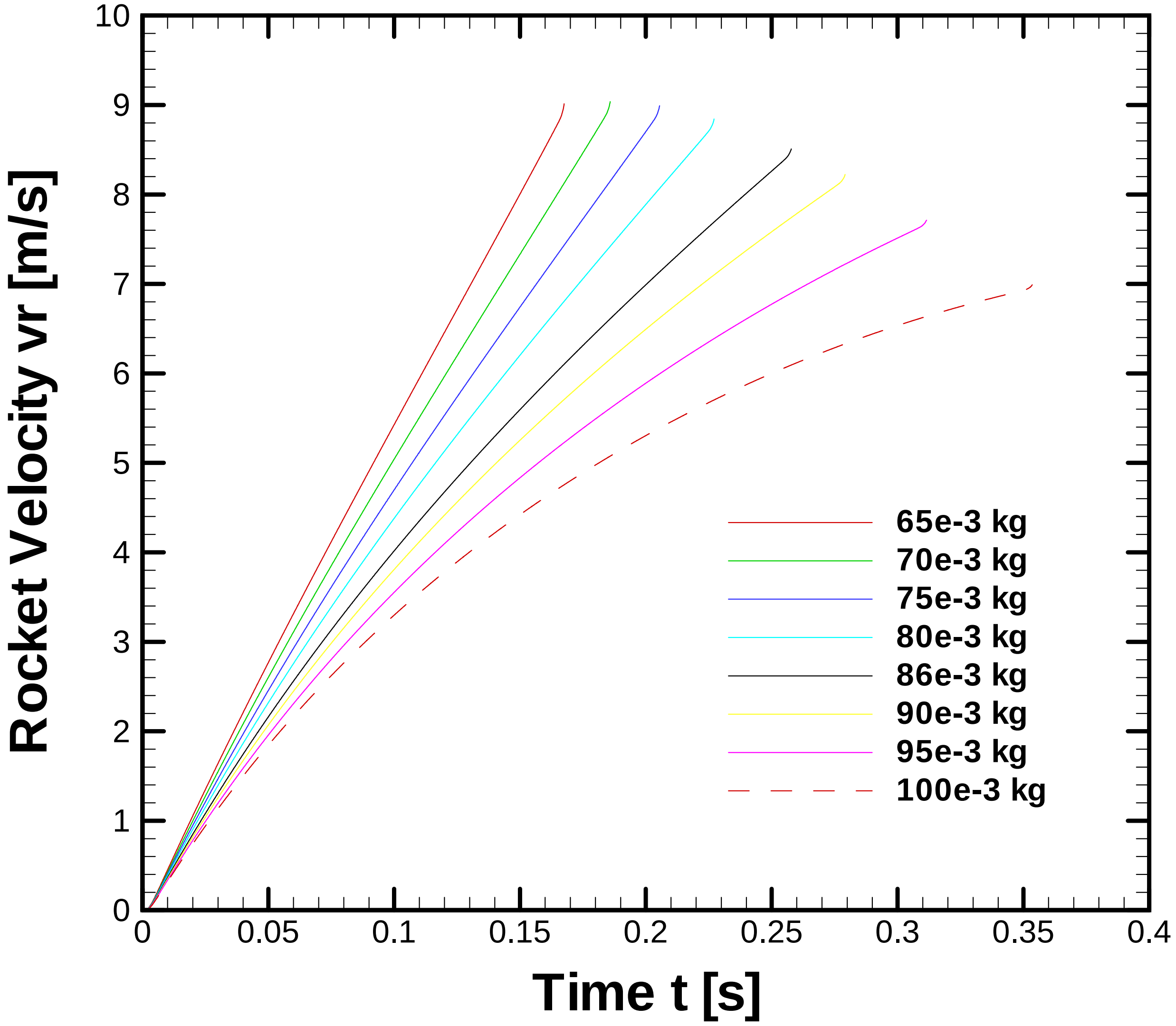

Fig. 3.3-II: Rocket Velocity versus time as function of start water mass

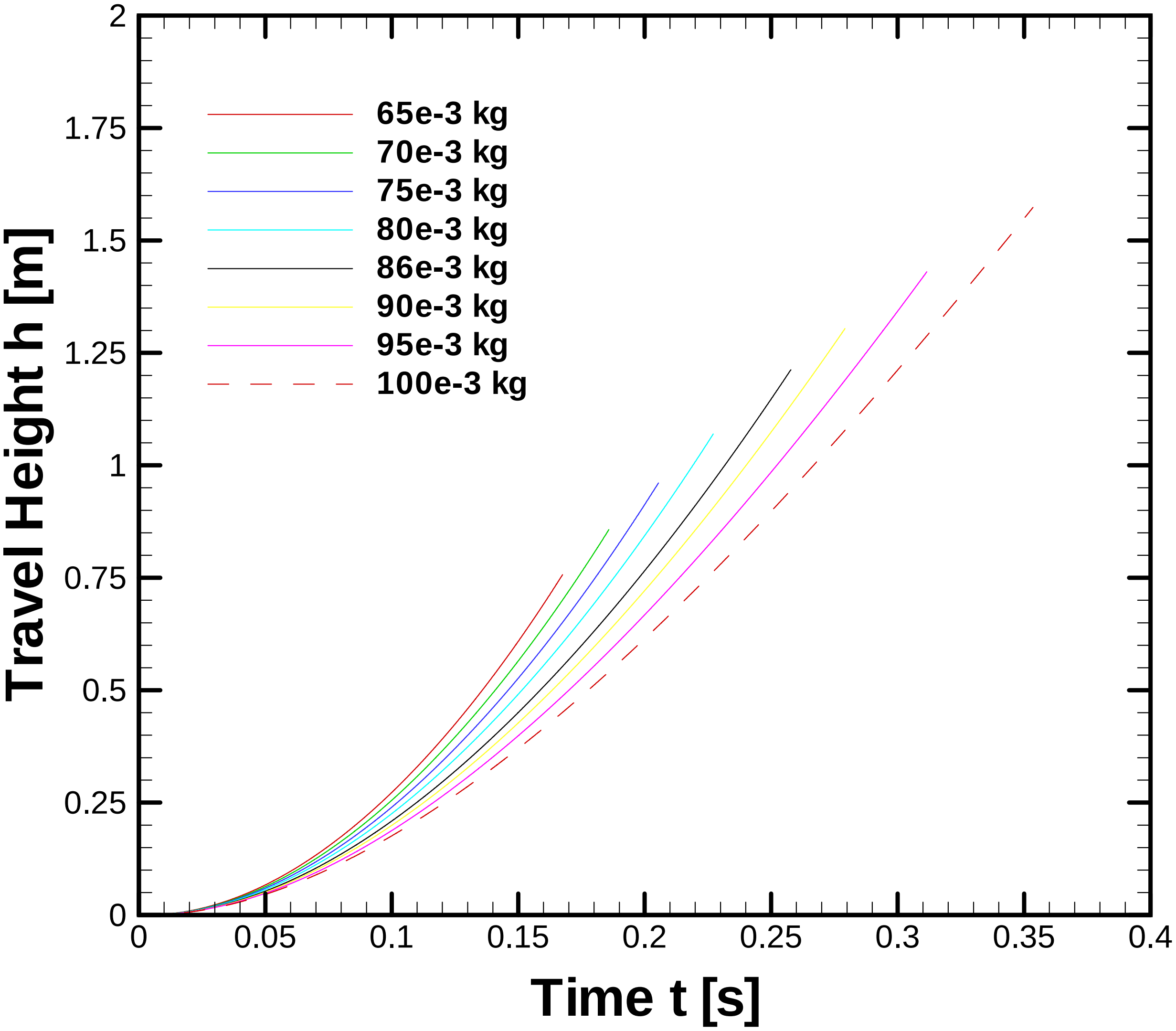

Fig. 3.3-III: Travel height versus time as function of start water mass

3.4 Nozzle Friction Coefficient Variation

The nozzle friction coefficient was varied from 0.0 to 1.0.

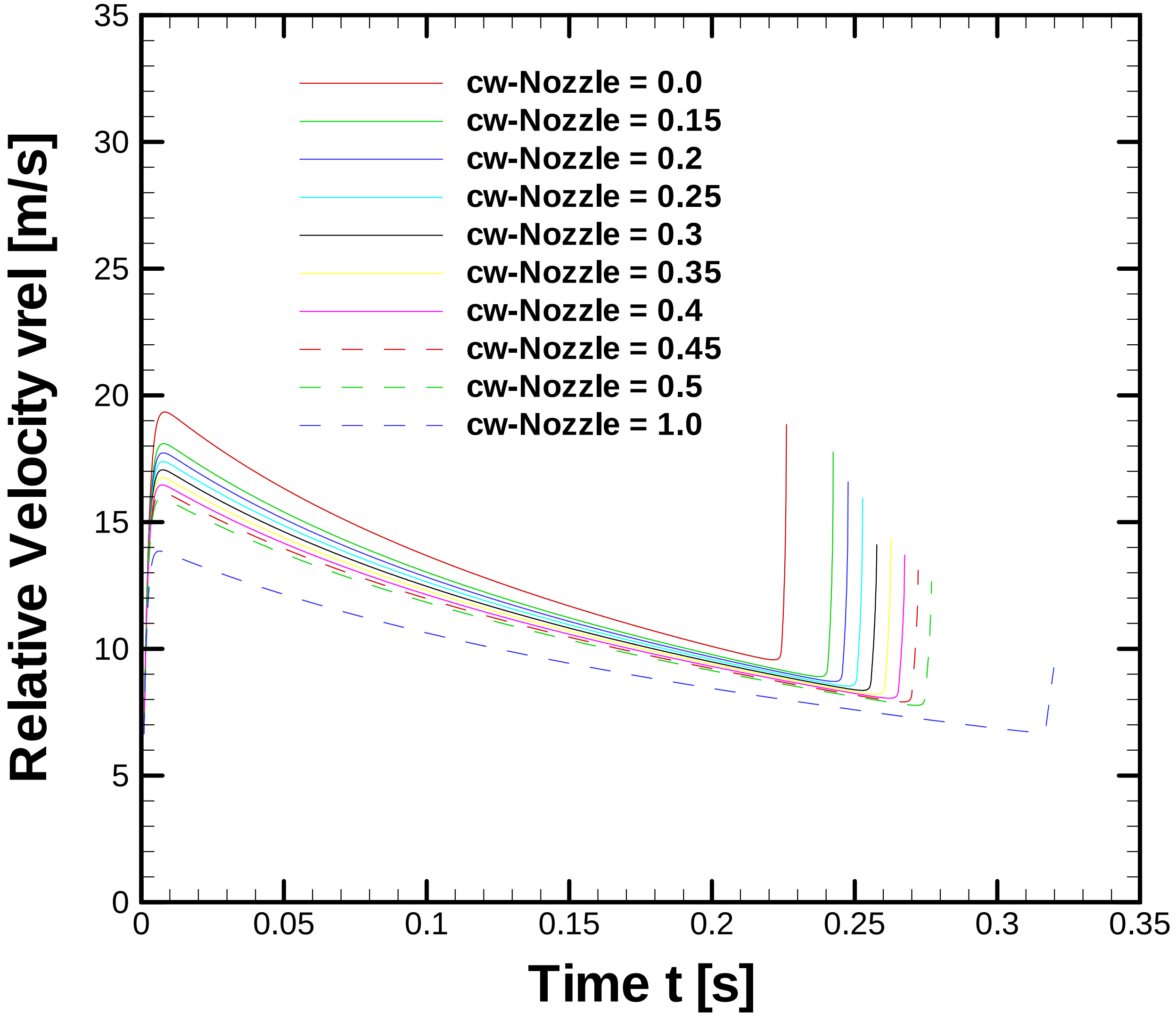

Fig. 3.4-I: Relative water velocity versus time as function of nozzle friction

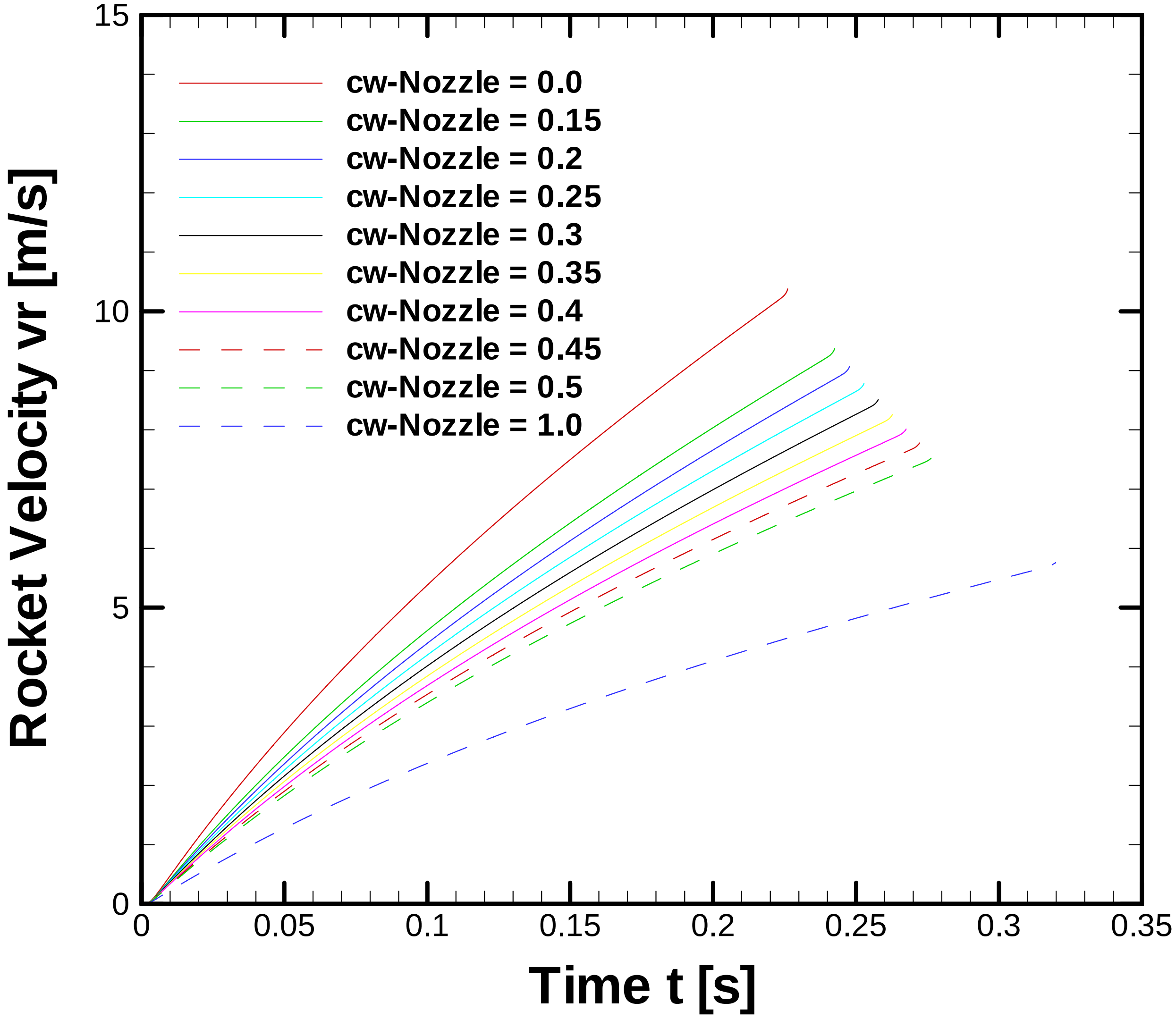

Fig. 3.4-II: Rocket velocity versus time as function of nozzle friction

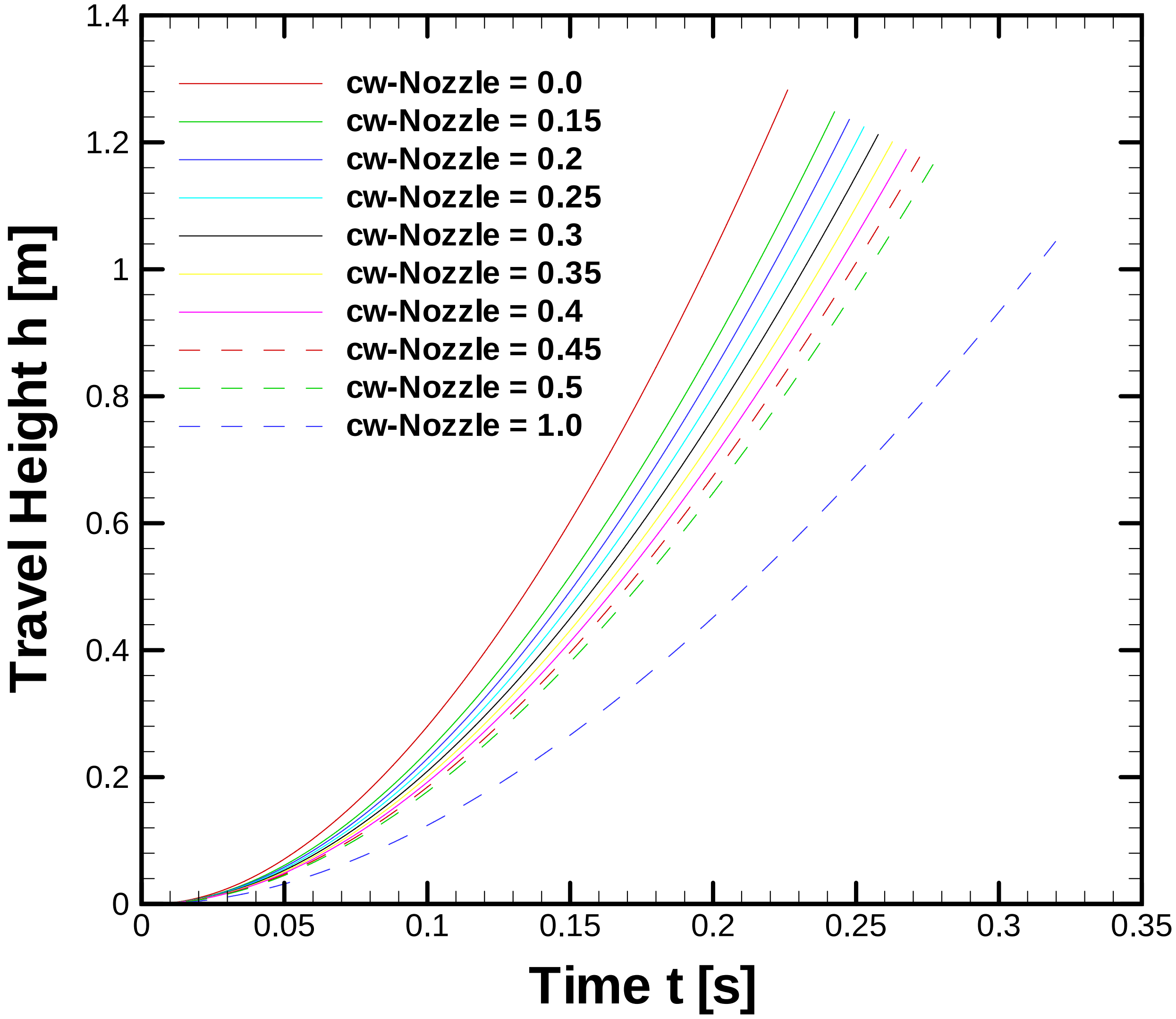

Fig. 3.4-III: Travel height versus time as function of nozzle friction

3.5 Polytropic Exponent Variation

The polytropic exponent was changed from 1.2 to 1.4.

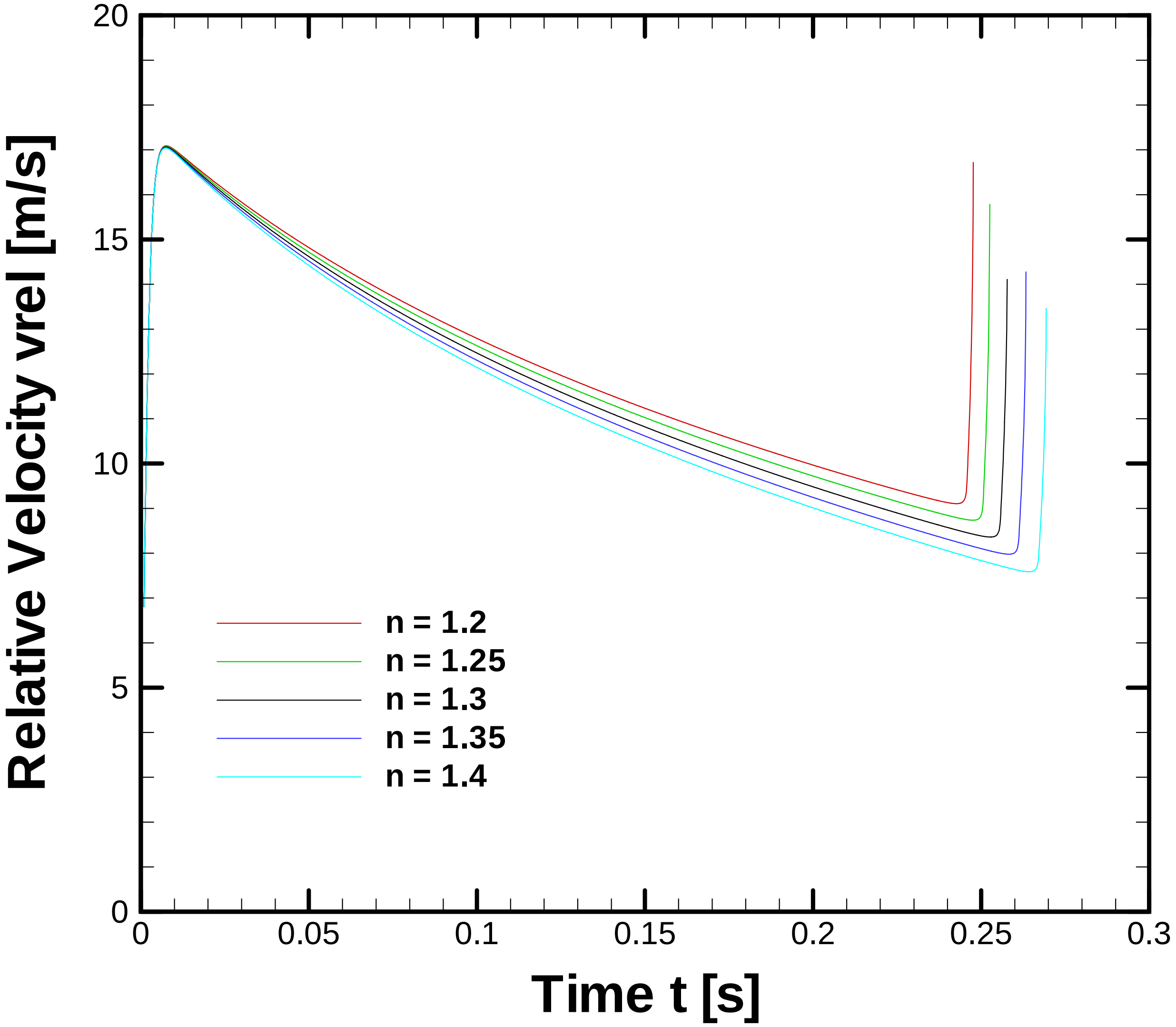

Fig. 3.5-I: Relative water velocity versus time as function of polytropic exponent

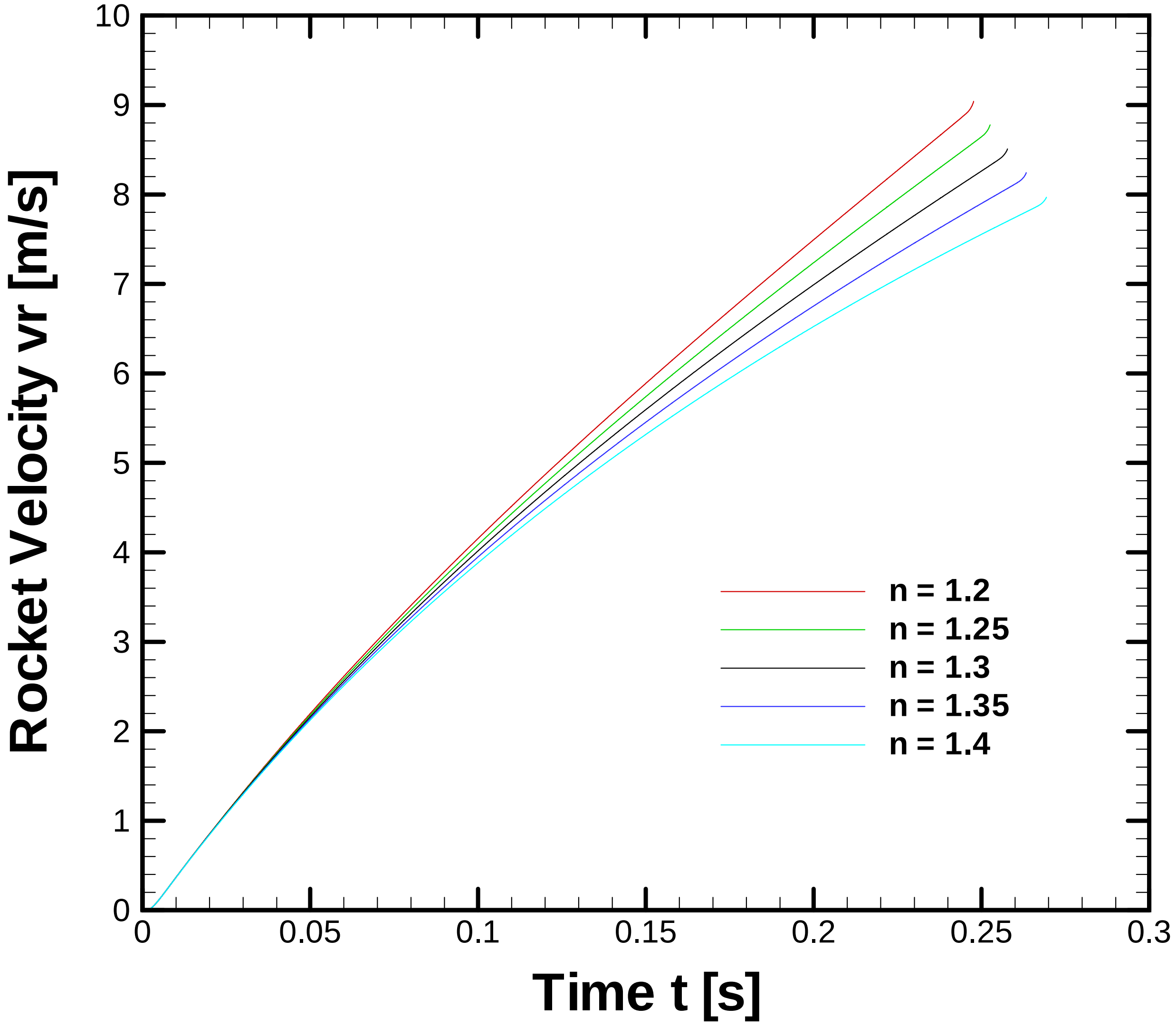

Fig. 3.5-I: Relative water velocity versus time as function of polytropic exponent

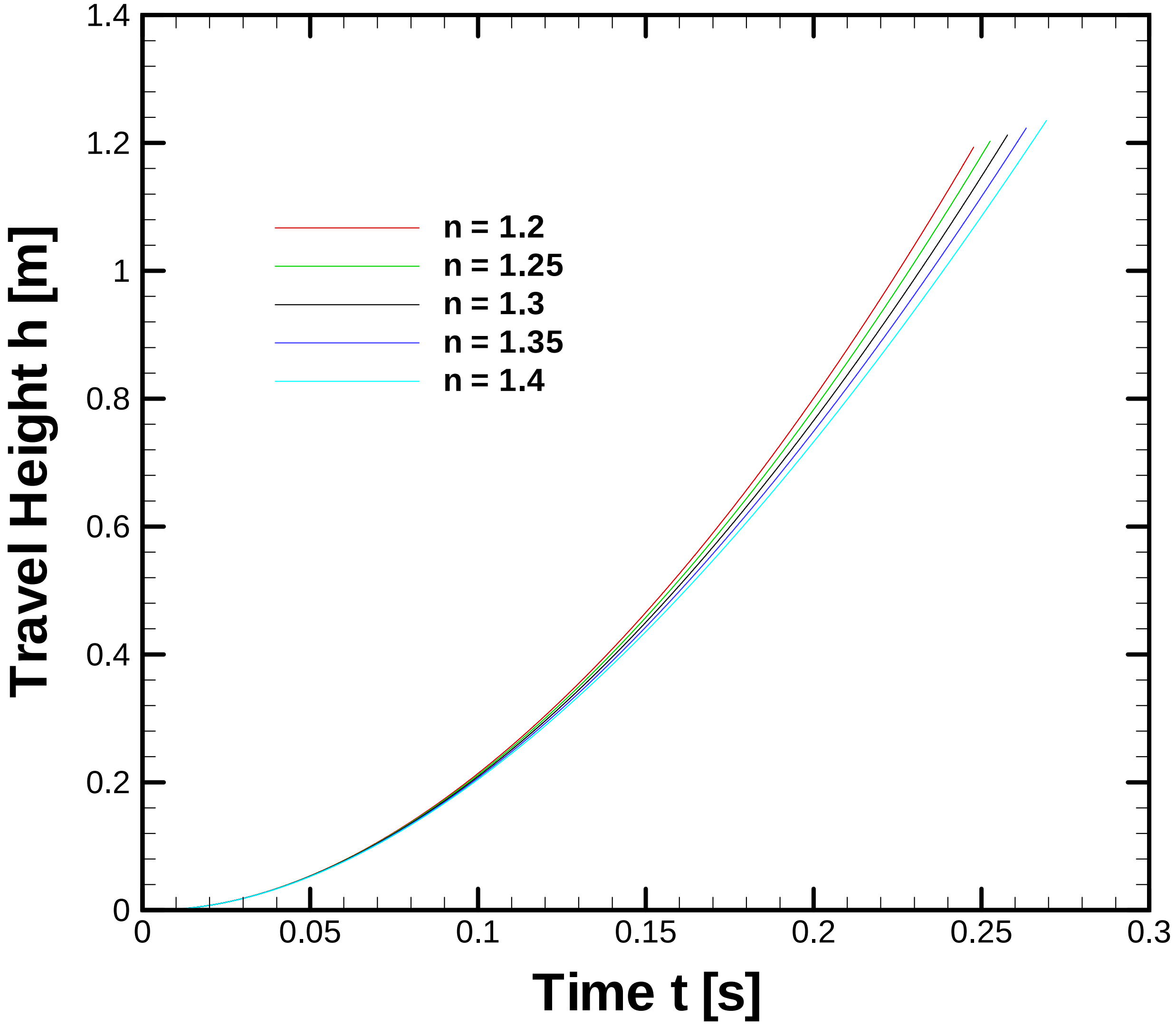

Fig. 3.5-I: Relative water velocity versus time as function of polytropic exponent

3.6 Convergence Investigation

Time step sweeps from Δt = 2e-5 till Δt = 2e-3 were performed. No solution dependency on the time step chosen occurred within this range. However, the solution became unstable for time steps larger or equal than Δt = 5e-3.

Grid sweeps from Δs = 2e-4 till Δs 2e-3 were performed. No solution dependency on the grid resolution chosen occurred. The algorithm remained stable even for very large values of Δs.

Appendix A References

[#]	Author	Titel
[1]	Adomeit G.	Unterlagen zur Vorlesung DYNAMIK II RWTH Aachen
[2]	Krause E.	Vorlesungen aus dem Aerodynamischen Institut STRÖMUNGSLEHRE I,II RWTH Aachen

Appendix B Program Code

```
Sub Rocket_Iteration()
'
' Rocket_Iteration Makro
' Makro am 12/28/00 von Dietzel aufgezeichnet
'
Sheets("vrel").Activate

'Definitions
ROH_W = Cells(3, 6)
ROH_A = Cells(4, 6)
CW = Cells(5, 6)
CW_R = Cells(6, 6)
P_AMB = Cells(7, 6)
P_ST = Cells(8, 6)
N = Cells(9, 6)
H_ST = Cells(10, 6)
DT = Cells(11, 6)
DS = Cells(12, 6)
VREL_ST = Cells(13, 6)
VR_0 = Cells(14, 6)
G = Cells(15, 6)
ETA_W = Cells(16, 6)
a = Cells(19, 6)
b = Cells(20, 6)
c = Cells(21, 6)
e = Cells(22, 6)
D = Cells(23, 6)
L = Cells(24, 6)
H_0 = Cells(25, 6)
V_0 = Cells(26, 6)
R_MAX = Cells(27, 6)
S_MAX = Cells(28, 6)
M_R = Cells(29, 6)
AD = Cells(31, 6)
A_MAX = Cells(32, 6)
V_ST = Cells(33, 6)
V_AIR_ST = Cells(34, 6)
C_POL = Cells(35, 6)
M_ST = Cells(36, 6)
M_0 = Cells(37, 6)
XREL_0 = Cells(38, 6)

i = 0
j = 1
```

```
k = 69
T = 0
VREL = 0
XREL = 0
VR = 0
VR2 = 0
HT = 0
H = H_ST

While H <= 0
  i = i + 1

  ' Acceleration Integral
  I_ACC = 0
  H2 = H
  While H2 <= 0
    If H2 < L Then
      GoSub 20
      I_ACC = I_ACC + AD / AR * DS
    Else
      I_ACC = I_ACC + AD / AD * DS
    End If
    H2 = H2 + DS
  Wend

  V1 = (-3.1415 * (1 / 7 * a ^ 2 * H ^ 7 + 1 / 3 * a * b * H ^ 6 + 1 / 5 * (2 * a * c + b ^ 2) * H ^ 5 +
                   1 / 4 * (2 * a * e + 2 * b * c) * H ^ 4 + 1 / 3 * (2 * b * e + c ^ 2) * H ^ 3 + c * e *
                   H ^ 2 + e ^ 2 * H - (1 / 7 * a ^ 2 * L ^ 7 + 1 / 3 * a * b * L ^ 6 + 1 / 5 * (2 * a * c
                   + b ^ 2) * L ^ 5 + 1 / 4 * (2 * a * e + 2 * b * c) * L ^ 4 + 1 / 3 * (2 * b * e + c ^
                   2) * L ^ 3 + c * e * L ^ 2 + e ^ 2 * L)))
  V2 = AD * -H
  If H <= L Then
    V = V1 + AD * (-L)
  Else
    V = V2
  End If

  V_AIR = V_0 - V

  'Relative Velocity
  H2 = H
  GoSub 20
  If H > L Then
    AR = AD
  End If
  P = C_POL * V_AIR ^ (-N) / 100000
```

```
    VREL = VREL + DT / (ROH_W * I_ACC) * ((P - P_AMB) * 100000 - 0.5 * ROH_W * (1 +
                                                  CW - (AD / AR) ^ 2) * VREL ^ 2)
    RAT_AD_AR = AD / AR

    'Momentum Conservation
    XREL = XREL + VREL * DT
    BOOST = (VREL ^ 2 / (XREL_0 - XREL) * DT)
    DRAG = 0.5 * CW_R * ROH_A / ROH_W * A_MAX / AD *
                                                  (VR2 ^ 2 / (XREL_0 - XREL)) * DT
    VR = VR - G * DT + BOOST
    VR2 = VR2 - G * DT + BOOST - DRAG
    If VR < 0 Then
      VR = 0
    End If
    If VR2 < 0 Then
      VR2 = 0
    End If
    HT = HT + VR2 * DT
    RE_D = D * VREL / ETA_W

    'Output
    Sheets("vrel").Activate
    Cells(k + i, 1) = T
    Cells(k + i, 2) = VREL
    Cells(k + i, 3) = VR
    Cells(k + i, 4) = VR2
    Cells(k + i, 5) = HT
    Cells(k + i, 6) = H
    Cells(k + i, 7) = DH
    Cells(k + i, 8) = I_ACC
    Cells(k + i, 9) = P
    Cells(k + i, 10) = V
    Cells(k + i, 11) = V_AIR
    Cells(k + i, 12) = RAT_AD_AR
    Cells(k + i, 13) = RE_D

    'Next Step
    DH = (AD / AR) * VREL * DT
    H = H + DH
    T = T + DT
    Cells(i, k + 7) = DH

    If i > 10000 Then GoTo 10

  Wend
```

```
 GoTo 10

'Subroutine Rocket Cross Section
20
R = a * H2 ^ 3 + b * H2 ^ 2 + c * H2 + e
AR = 3.1415 * R ^ 2
Return

10
'
End Sub
```